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Unbounded stratified flow over a vertical barrier

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The horizontal, linearly stratified, non-diffusive, high Reynolds number flow of an unbounded fluid over a two-dimensional vertical barrier is studied for a range of internal Froude numbers (F_i) under the Oseen and Boussinesq approximations. For $F_i > 0{\cdot}47$ the most prominent feature of the flow is the system of large amplitude lee waves located downstream of the barrier with crests tilted in the upstream direction. For $0{\cdot}47 < F_i < 0{\cdot}6$ the crests actually extend upstream of the barrier and appear as flows of alternating direction over the barrier. For $0{\cdot}47 < F_i < 0{\cdot}5$ reversed flows due to these waves actually extend far upstream. For $F_i < 0{\cdot}47$ a blocking column upstream of the obstacle, as well as large amplitude lee waves, is present. For even smaller F_i the amplitude of the lee-wave system diminishes but the blocking column remains. It is also shown that the steady-state solution obtained by Trustrum (1964, 1971) for the density and pressure field is drastically altered if a small viscosity is retained in the transient analysis.

1. Introduction

Stably stratified flows over obstacles are of some geophysical interest. High Reynolds number flows, i.e. inviscid flows, have been studied analytically in several manners. The first approach is based upon the use of Long's (1953) equation, the nonlinear vorticity equation integrated along streamlines. If the upstream flow is assumed to be linearly stratified with $\rho U^2 =$ constant, Long's equation becomes linear. Owing to this fact and to the relative ease with which these conditions are (almost) achieved in the laboratory and observed in nature, this linear equation has been used extensively to study inviscid stratified flows over obstacles in channels.

However, at low internal Froude numbers, disturbances can propagate far upstream, thus negating the assumption of (almost) uniform flow there which is required in Long's linear model. For example, Bretherton (1967) considered an initial-value problem for two-dimensional flow over a cylinder in an unbounded stratified flow for $F_i = 0$. He found blocking columns both upstream and downstream of the cylinder with horizontal flow above and below the cylinder.

Miles (1971) has shown, starting with Long's equation, that reversed flow upstream of a vertical barrier in a half-space first occurs (at $x = y = 0$) when F_i is lowered to $1/2{\cdot}05$. For a further discussion see § 5.

Trustrum (1971) considered the inviscid flow past a barrier in a channel of finite height using the Oseen and Boussinesq approximations. The general form of her steady-state solution was obtained in an earlier paper (Trustrum 1964) from an inviscid transient analysis. We note here that the general solution which we obtain in this paper from a steady slightly viscous analysis differs in form from Trustrum's solution.

In an earlier paper, the author (1968, 1971) considered the low Reynolds number stratified flow past a vertical barrier in an unbounded region and found a blocking column upstream of the cylinder, uniform flow downstream of the barrier, and a vertical shear layer engulfing the plate.

The purpose of this work is to obtain the overall picture of the high Reynolds number, non-diffusive, stratified flow over a vertical barrier in an unbounded region over the range of F_i of general interest. The Oseen model will be adopted. Since reversed flows will occur for some F_i, Long's equation could not be used and the theoretical alternative to the Oseen model is to solve the full Navier–Stokes equations in an unbounded region. Solutions for the stream function will be obtained upstream and downstream of the barrier and matched appropriately across the plane of the barrier.

2. Formulation of the problem

A vertical barrier of height $2b$ moves horizontally with speed U through an initially linearly stratified ($\rho' = \rho_0(1-\beta z')$) non-diffusive fluid. The flow as viewed from the frame of reference of the barrier is steady and the disturbance to uniform flow is limited by viscosity to some finite (though large) region of space, see figure 1.

The dimensional governing equations under the Boussinesq approximation for steady, two-dimensional, non-diffusive flow are as follows:

$$\rho_0(\mathbf{v}'.\nabla')\,u' = -\partial p'/\partial x' + \mu\nabla'^2 u', \tag{1a}$$

$$\rho_0(\mathbf{v}'.\nabla')\,w' = -\partial p'/\partial z' - \rho' g + \mu\nabla'^2 w', \tag{1b}$$

$$\partial u'/\partial x' + \partial w'/\partial z' = 0, \tag{1c}$$

or

$$u' = -\partial\overline{\psi}'/\partial z', \quad w' = \partial\overline{\psi}'/\partial x' \tag{1d}$$

and

$$\rho' = \rho'(\overline{\psi}'). \tag{1e}$$

Far upstream, where viscosity has eliminated the disturbance to uniform flow, $\overline{\psi}' \to -Uz'$ and $\rho' \to \rho_0(1-\beta z')$. Hence,

$$\rho' = \rho_0[1 + (\beta/U)\,\overline{\psi}']. \tag{2}$$

We obtain the vorticity equation by cross-differentiation of (1a) and (1b). We then non-dimensionalize velocities by U, distances by b and the stream function by Ub. We obtain

$$(\mathbf{v}.\nabla)\,\nabla^2\overline{\psi} + (1/F_i^2)\,\partial\overline{\psi}/\partial x - (1/Re)\,\nabla^4\overline{\psi} = 0, \tag{3}$$

where $F_i^2 = U^2/\beta g b^2$ and $Re = Ub/\nu$. We model the convective operator with the Oseen operator, i.e. we replace $(\mathbf{v}.\nabla)$ by $\partial/\partial x$. We also write

$$\overline{\psi} = -z + \psi, \tag{4}$$

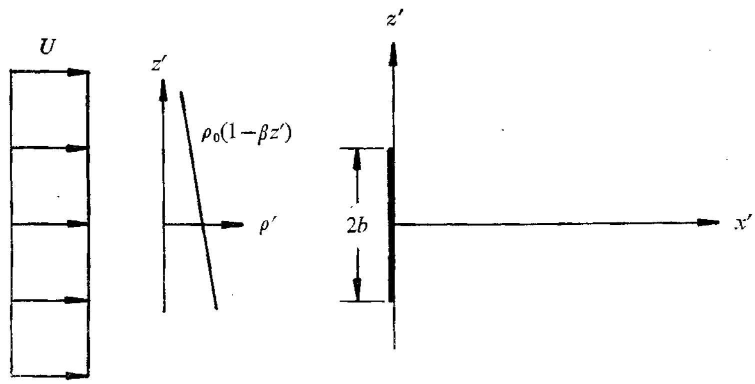


FIGURE 1. Geometry of the flow field.

and obtain

$$\frac{\partial}{\partial x}(\nabla^2\psi)+\frac{1}{F_i^2}\frac{\partial\psi}{\partial x}-\frac{1}{Re}\nabla^4\psi = 0. \tag{5}$$

The solutions we obtain to (5) show regions of reversed flow. We must then ask how these solutions are related to the solutions of (3). We cannot answer this question directly. The best we could do is, using the solutions we shall obtain, compute $(\mathbf{v}.\nabla)\ \nabla^2\psi$ and compare this with $\partial(\nabla^2\psi)/\partial x$. The computation would show that at some points along any streamline the nonlinear operator exceeds the linear operator in magnitude and at other points the reverse is true. Hence, the overall effect on the streamline pattern is not clear. The best we can hope for is that the solutions to (5) qualitatively model the solutions to (3). However, the analytical alternative to the Oseen approach is the numerical solution of (3) for an infinite domain, a task considerably more difficult than that undertaken here. We now adopt equation (5) as our model and determine what it tells us. We shall solve this equation for $z \geqslant 0$, since the flow is symmetric with respect to the x axis.

The boundary conditions imposed upon ψ are as follows:

$$\psi \to 0 \quad \text{as} \quad |x,z| \to \infty; \tag{6a}$$

$$\psi = z, \quad \partial\psi/\partial x = 0 \quad \text{at} \quad x = 0 \quad \text{for} \quad z \leqslant 1; \tag{6b,c}$$

$$\psi, \quad \frac{\partial\psi}{\partial x}, \quad \frac{\partial^2\psi}{\partial x^2}, \quad \frac{\partial^3\psi}{\partial x^3} \quad \text{continuous at } x = 0 \text{ for } z > 1. \tag{6d}$$

We shall solve (5) subject to (6) for $Re \to \infty$ for a range of F_i.

3. Method of solution

To solve the problem we shall proceed as follows. We first obtain the general solution to (5) in terms of the Fourier sine transforms of $\psi(0,z)$ $(F(k))$ and $\partial\psi(0,z)\,\partial x$ $(G(k))$. The requirement that $\partial^2\psi/\partial x^2$ and $\partial^3\psi/\partial x^3$ be continuous across $x = 0$ for $z > 1$ will lead to two coupled integral equations relating $F(k)$ and $G(k)$ for $z > 1$. $F(k)$ and $G(k)$ will then be obtained from these equations in terms of the

Fourier sine transforms of two functions of z, $R_1(z)$ and $R_2(z)$, which are identically zero for $z > 1$. Conditions (6b) and (6c) then lead to integral equations for the Fourier sine transforms of $R_1(z)$ ($R_1(k)$) and $R_2(z)$ ($R_2(k)$) on $0 \leqslant z \leqslant 1$. These equations are finally solved approximately through the use of Fourier sine series on $0 \leqslant z \leqslant 1$.

Let us then begin by defining the Fourier sine transforms of the upstream and downstream solutions for ψ, $H_U(k,x)$ and $H_D(k,x)$ respectively. For $x \leqslant 0$, $x \geqslant 0$

$$\psi(x,z) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty H_{U,D}(k,x) \sin kz \, dk. \tag{7a,b}$$

If we substitute these equations into (5), and then satisfy the requirement that ψ and $\partial\psi/\partial x$ be continuous across $x = 0$, we find

$$H_{U,D}(k,x) = \frac{\alpha_{4,2}(k)\,F(k) - G(k)}{\alpha_{4,2}(k) - \alpha_{3,1}(k)}\, e^{\alpha_{3,1}(k)x} + \frac{G - \alpha_{3,1}F}{\alpha_{4,2} - \alpha_{3,1}}\, e^{\alpha_{4,2}x}. \tag{8a,b}$$

The $\alpha_i(k)$ are the four roots of

$$O = \alpha_i(\alpha_i^2 - k^2) + (1/F_i^2)\,\alpha_i - (1/Re)\,(\alpha_i^2 - k^2)^2 \tag{8c}$$

and

$$F(k) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \psi(0,z) \sin kz \, dz, \tag{8d}$$

$$G(k) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\partial\psi}{\partial x}(0,z) \sin kz \, dz. \tag{8e}$$

The roots α_1 and α_2 are complex conjugates for $0 \leqslant kF_i < 1$ and have negative real parts for all k which increase in magnitude monotonically with k. The roots α_3 and α_4 are real and positive and increase monotonically with k.

Furthermore, for $kF_i < 1$, the real parts of α_1, α_2 and α_3 go to zero as $Re \to \infty$. For $kF_i > 1$, α_1 goes to zero (for any fixed k) as $Re \to \infty$. For $Re \to \infty$ for fixed k, the roots are as follows. For $0 \leqslant kF_i < 1$

$$\alpha_{1,2} = \pm i(\alpha^2 - k^2)^{\frac{1}{2}}, \quad \alpha_3 = 0, \quad \alpha_4 = Re. \tag{9a,b,c,d}$$

For $kF_i > 1$,

$$\alpha_1 = 0, \quad \alpha_{2,3} = \mp (k^2 - \alpha^2)^{\frac{1}{2}}, \quad \alpha_4 = Re, \tag{9e,f,g,h}$$

where $\alpha \equiv 1/F_i$. We note that α_4 represents a boundary layer of thickness Re^{-1} just upstream of the plate. For Re finite but large, and for

$$0 \leqslant kF_i < 1 - (F_i Re)^{-\frac{2}{3}},$$

$$\alpha_4 = \frac{1}{F_i^2 Re}\left(\frac{(kF_i)^4}{1 - (kF_i)^2}\right). \tag{10}$$

From (10), we can conclude that the upstream disturbance will decay at distances of order $F_i^2 Re$.

The general solution of (8b) differs from Trustrum's (1971) in one important aspect. In her downstream solution, for $0 \leqslant kF_i \leqslant 1$, she has a component independent of x, while our downstream solution is oscillatory in x for $0 \leqslant kF_i < 1$.

To satisfy the requirement that $\partial^2\psi/\partial x^2$ and $\partial^3\psi/\partial x^3$ be continuous across

$x = 0$, we substitute (8) into (7), differentiate with respect to x, and find that for $z > 1$

$$\int_0^\infty [(-\alpha_3\alpha_4 + \alpha_1\alpha_2)F(k) + (\alpha_3+\alpha_4-\alpha_1-\alpha_2)G(k)]\sin kz\,dk = 0, \tag{11}$$

$$\int_0^\infty [-\alpha_3\alpha_4(\alpha_3+\alpha_4)F(k) + (\alpha_4^2+\alpha_3\alpha_4+\alpha_3^2-\alpha_1^2-\alpha_1\alpha_2-\alpha_2^2)G(k)]\sin kz\,dk = 0. \tag{12}$$

We use the roots given in (9) in the above equations to obtain

$$\int_0^\infty \{Re[-\alpha_3 F + G] + [\alpha_1\alpha_2 F + (\alpha_3-\alpha_1-\alpha_2)G]\}\sin kz\,dk = 0, \tag{13}$$

$$\int_0^\infty \{Re^2[-\alpha_3 F + G] + Re[-\alpha_3^2 F + \alpha_3 G] + [(\alpha_3^2-\alpha_1^2-\alpha_2^2-\alpha_1\alpha_2)G]\} \times \sin kz\,dk = 0. \tag{14}$$

We now expand $F(k)$ and $G(k)$ as follows:

$$\left.\begin{matrix} F(k) \\ G(k) \end{matrix}\right\} = \sum_{n=0}^{\infty} Re^{-n} \left\{\begin{matrix} F_n(k), & (15) \\ G_n(k), & (16) \end{matrix}\right.$$

Upon substituting these expansions into (13) and (14) and equating the coefficients of Re^1 in (13) and Re^2 in (14), we obtain for $z > 1$

$$\int_0^\infty [-\alpha_3 F_0 + G_0]\sin kz\,dk = 0. \tag{17}$$

Equating the coefficients of Re^0 in (13), we obtain for $z > 1$

$$\int_0^\infty [-\alpha_3 F_1 + G_1]\sin kz\,dk = -\int_0^\infty [\alpha_1\alpha_2 F_0 + (\alpha_3-\alpha_1-\alpha_2)G_0]\sin kz\,dk. \tag{18}$$

By equating the coefficients of Re^1 in (14), we obtain for $z > 1$

$$\int_0^\infty [-\alpha_3 F_1 + G_1]\sin kz\,dk = -\int_0^\infty [-\alpha_3^2 F_0 + \alpha_3 G_0]\sin kz\,dk. \tag{19}$$

Equations (18) and (19) then imply that

$$\int_0^\infty [(\alpha_1\alpha_2+\alpha_3^2)F_0 - (\alpha_1+\alpha_2)G_0]\sin kz\,dk = 0. \tag{20}$$

Equations (17) and (20) are two coupled integral equations for $F(k)$ and $G(k)$ for $z > 1$. Using the roots given in (9), the requirement that the disturbance decays as $z \to \infty$, and dropping the zero subscripts, we obtain for $z > 1$

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty [G - \alpha_3 F]\sin kz\,dk = 0, \tag{21a}$$

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty [\{1-2U(k-\alpha)\}F]\sin kz\,dk - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_\alpha^\infty \frac{G\sin kz\,dk}{(k^2-\alpha^2)^{\frac{1}{2}}} = 0, \tag{21b}$$

where $\alpha = 1/F_i$. Equations ($6b, c$) imply that for $z \leqslant 1$

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty F\sin kz\,dk = z, \tag{21c}$$

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty G\sin kz\,dk = 0. \tag{21d}$$

The solution of equations (21) constitutes the crux of our problem. The difficulty lies in (21a, b), which we now rewrite for all z as follows:

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty [G-\alpha_3 F]\sin kz\,dk = 2R_1(z), \qquad (21a')$$

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty [1-2U(k-\alpha)]F\sin kz\,dk - \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_\alpha^\infty \frac{G}{(k^2-\alpha^2)^{\frac{1}{2}}}\sin kz\,dk = 2R_2(z), \qquad (21b')$$

where $R_1(z)$ and $R_2(z)$ are, thus far, arbitrary functions of z which vanish identically for $z > 1$.

We define the Fourier sine transforms of $R_1(z)$ and $R_2(z)$ as follows:

$$R_{1,2}(k) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^1 R_{1,2}(z)\sin kz\,dz. \qquad (22a,b)$$

We may therefore solve for $F(k)$ and $G(k)$ in terms of $R_1(k)$ and $R_2(k)$ since (21) holds for all z. We find for $k < \alpha$

$$G(k) = 2R_1(k), \quad F(k) = 2R_2(k) \qquad (23a,b)$$

and for $k > \alpha$

$$G(k) = R_1(k) - (k^2-\alpha^2)^{\frac{1}{2}} R_2(k), \qquad (23c)$$

$$F(k) = -R_2(k) - R_1(k)/(k^2-\alpha^2)^{\frac{1}{2}}. \qquad (23d)$$

Substituting this into (8) with $x = 0$, we find for $z \leqslant 1$

$$-R_2(z) + 3\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\alpha R_2(k)\sin kz\,dk - \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_\alpha^\infty \frac{R_1 \sin kz\,dk}{(k^2-\alpha^2)^{\frac{1}{2}}} = z, \qquad (24a)$$

$$R_1(z) + \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\alpha R_1(k)\sin kz\,dk - \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_\alpha^\infty (k^2-\alpha^2)^{\frac{1}{2}} R_2 \sin kz\,dk = 0. \qquad (24b)$$

Substituting (23) into (8) we find for $x < 0$

$$\psi(x,z) = 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\alpha R_2 \sin kz\,dk - \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_\alpha^\infty \left[R_2 + \frac{R_1}{(k^2-\alpha^2)^{\frac{1}{2}}}\right] e^{(k^2-\alpha^2)^{\frac{1}{2}}x}\sin kz\,dk \qquad (25a)$$

and for $x \geqslant 0$

$$\psi(x,z) = -2R_2(z) + 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\alpha R_2 \sin kz\,dk + 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\alpha \left[R_2 \cos[(\alpha^2-k^2)^{\frac{1}{2}}x] + \frac{R_1 \sin[(\alpha^2-k^2)^{\frac{1}{2}}x]}{(\alpha^2-k^2)^{\frac{1}{2}}}\right]\sin kz\,dk - \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_\alpha^\infty \left[\frac{R_1}{(k^2-\alpha^2)^{\frac{1}{2}}} - R_2\right] e^{-(k^2-\alpha^2)^{\frac{1}{2}}x}\sin kz\,dk. \qquad (25b)$$

The functions $R_1(k)$ and $R_2(k)$ still remain to be determined. We first write

$$R_{1,2}(z) = \sum_{n=1}^{\infty} r_{1n,2n}\sin n\pi z. \qquad (26a,b)$$

Equations (22) then imply that

$$R_{1,2}(k) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin k \sum_{n=1}^{\infty} \frac{(-1)^n n\pi}{k^2-(n\pi)^2} r_{1n,2n}. \qquad (26c,d)$$

We substitute (26) into (24) and multiply through by $\sin m\pi z$ and then integrate with respect to z from zero to one. This leads to the following set of equations for $m = 1, 2, 3, \ldots$.

$$-\sum_{n=1}^{\infty} I_{mn} r_{1n} + \sum_{n=1}^{\infty} (-\tfrac{1}{2}\delta_{nm} + 3J_{mn}) r_{2n} = (-1)^{m+1}/m\pi, \tag{27a}$$

$$\sum_{n=1}^{\infty} (\tfrac{1}{2}\delta_{mn} + J_{mn}) r_{1n} - \sum_{n=1}^{\infty} K_{mn} r_{2n} = 0, \tag{27b}$$

where

$$I_{mn} = \frac{2}{\pi}(-1)^{m+n}(n\pi)(m\pi)\int_{\alpha}^{\infty} \frac{\sin^2 k\, dk}{(k^2-\alpha^2)^{\frac{1}{2}}(k^2-(n\pi)^2)(k^2-(m\pi)^2)}, \tag{27c}$$

$$J_{mn} = \frac{2}{\pi}(-1)^{m+n}(n\pi)(m\pi)\int_{0}^{\alpha} \frac{\sin^2 k\, dk}{(k^2-(n\pi)^2)(k^2-(m\pi)^2)}, \tag{27d}$$

$$K_{mn} = \frac{2}{\pi}(-1)^{m+n}(n\pi)(m\pi)\int_{\alpha}^{\infty} \frac{(k^2-\alpha^2)^{\frac{1}{2}}\sin^2 k\, dk}{(k^2-(n\pi)^2)(k^2-(m\pi)^2)}. \tag{27e}$$

To solve (27), we truncate after M terms, i.e. we assume that $r_{1m}, r_{2m} = 0$ for $m > M$ and solve (27a, b) for $m = 1, 2, \ldots, M$. We then substitute into (24) to check the accuracy of our truncation. The integrals that we required were computed using Simpson's rule with varying step size on a Univac 1108. We note that, since for large k (where neglected viscous effects come into play) $G \to \sin k/k$, the vertical velocity becomes unbounded as $\ln(1-z)$ as $z \to 1^-$. Furthermore, since we are approximating z, R_1 and R_2 by finite sine series (6b, c) cannot be satisfied for $z \to 1^-$. Hence, there will be some error in the neighbourhood of $z \lesssim 1$ on the downstream side of the barrier, but this does not prevent us from obtaining good results elsewhere.

4. Numerical results

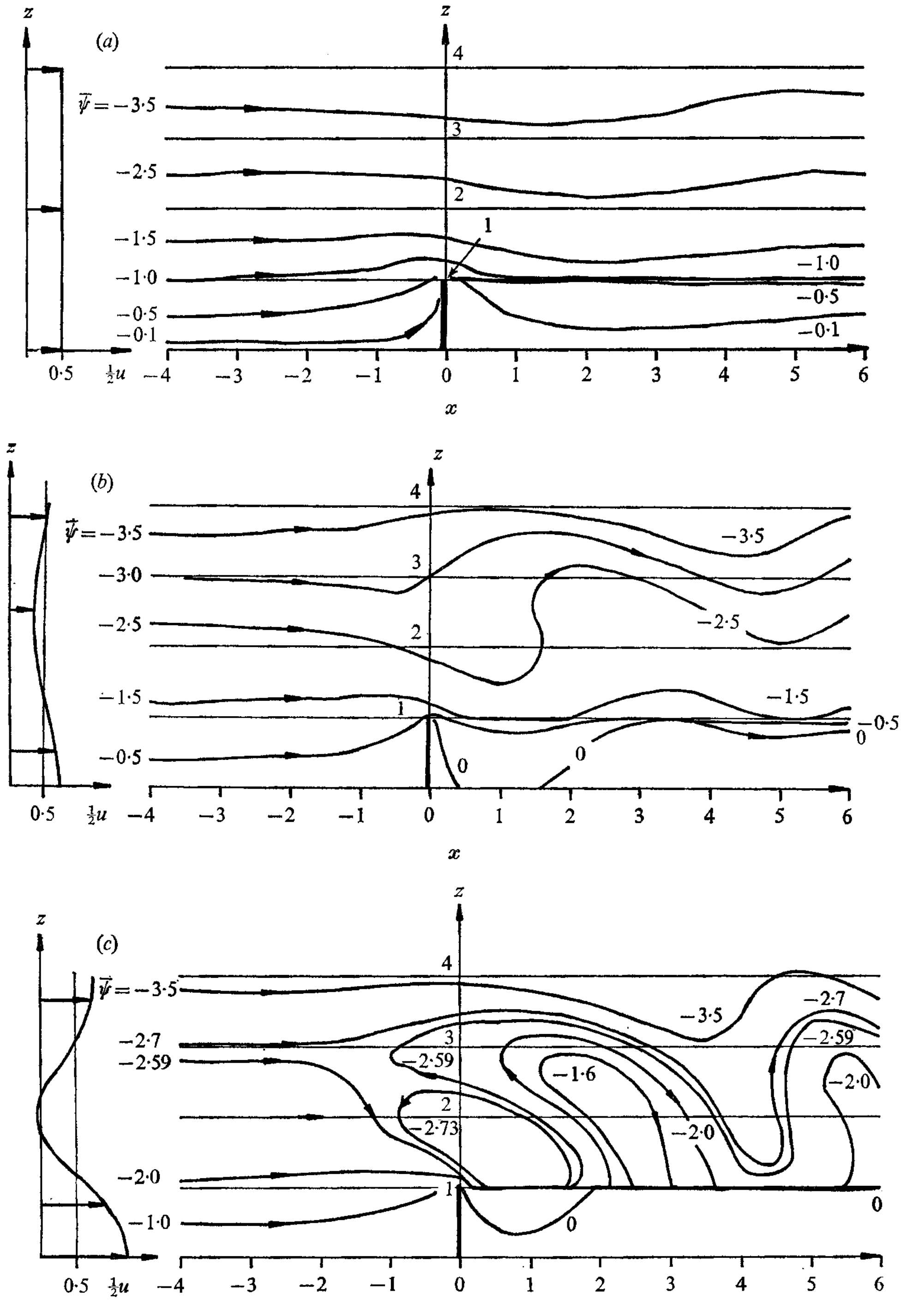
For the cases considered, we generally chose $M = 15$ or 18. This led to error in (6b), oscillatory in z of order 2 or 3 % for $z < 1$, well in keeping with the qualitative description afforded, at best, by the Oseen approximation. The errors in $\partial\psi(0, z)/\partial x$, $0 \leqslant z < 1$ are of order 0·1 and are again oscillatory in z. This leads to some error in the immediate vicinity of the body, but small errors elsewhere.

In figures 2 (a)–(g), we plot the streamline pattern and the far-upstream velocity profile

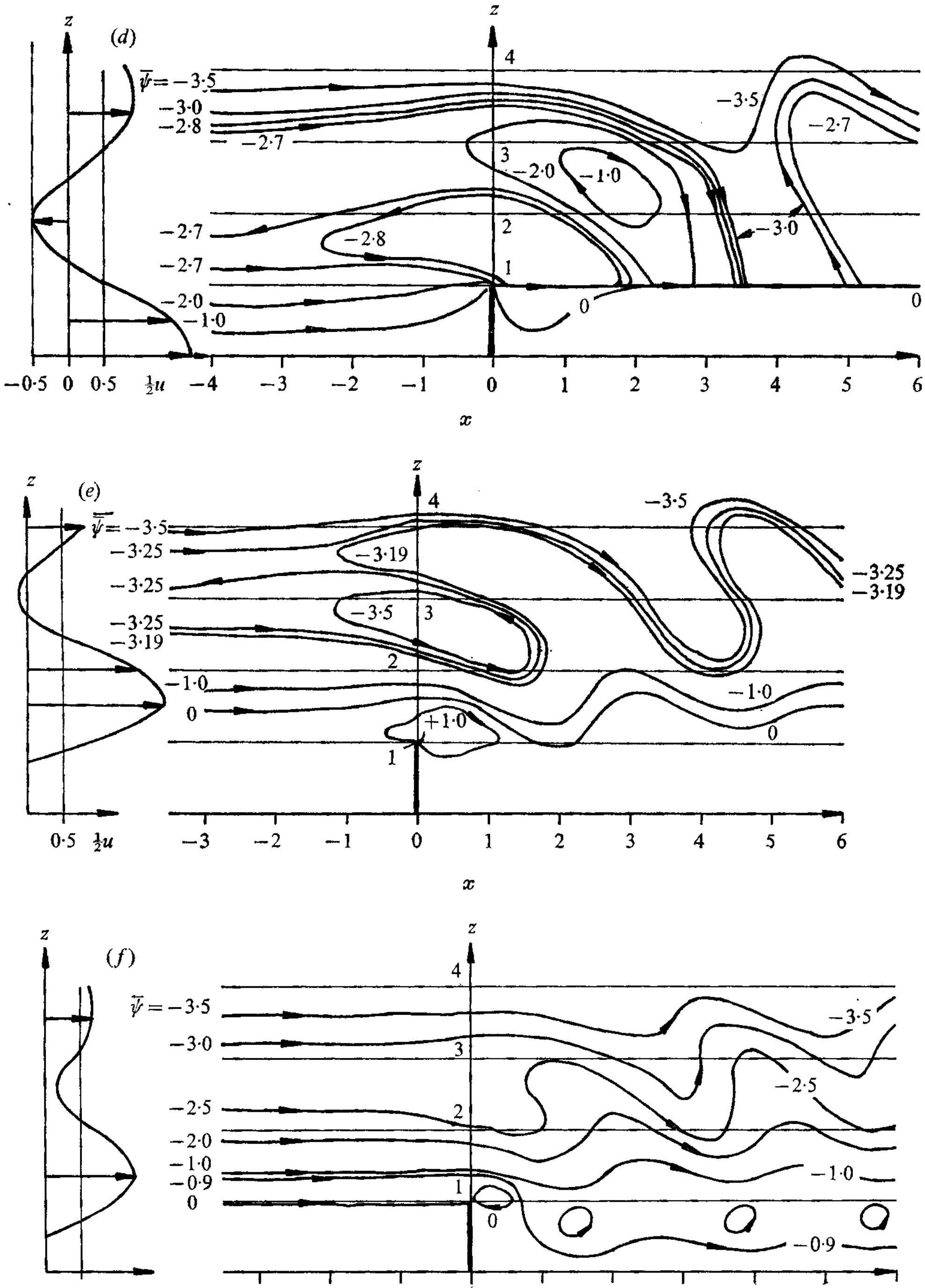
$$\left(1 - 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\alpha} k \cos kz\, R_2(k)\, dk\right)$$

for a number of values of F_i ranging from $F_i = 1·0$ ($\alpha = 1·0$) down to $F_i = 0·285$ ($\alpha = 3·5$). We shall discuss our results by comparing wake regions, lee-wave patterns and blocking columns.

First we shall discuss the wake region, i.e. $x > 0$ and $0 \leqslant z \leqslant 1$. For $\alpha = 1·0$, the flow is in the downstream direction with speeds about half the free-stream value. For $\alpha = 1·7$ a separated bubble has formed. For $\alpha = 2·0$, 2·1 and 2·5, the entire wake has separated and for $\alpha = 2·5$ has become attached to the upstream blocking column. For $\alpha = 3·0$ and 3·5, the wake consists of a series of



FIGURES 2 (*a*), (*b*) and (*c*). For legend see page 384.



FIGURES 2 (*d*), (*e*) and (*f*). For legend see page 384.

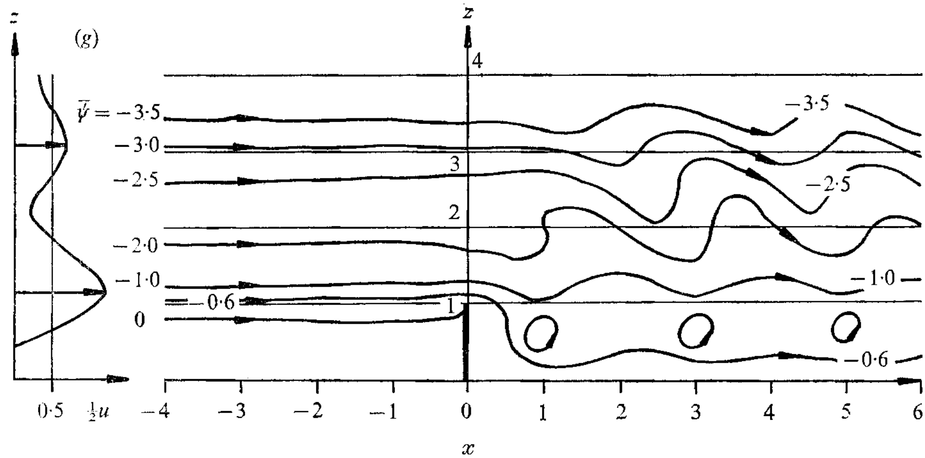


FIGURE 2. Streamline pattern for (a) $F_i = 1{\cdot}00$ ($\alpha = 1{\cdot}0$), (b) $F_i = 0{\cdot}588$) ($\alpha = 1{\cdot}7$), (c) $F_i = 0{\cdot}500$ ($\alpha = 2{\cdot}0$), (d) $F_i = 0{\cdot}476$ ($\alpha = 2{\cdot}1$), (e) $F_i = 0{\cdot}400$ ($\alpha = 2{\cdot}5$), (f) $F_i = 0{\cdot}333$ ($\alpha = 3{\cdot}0$) and (g) $F_i = 0{\cdot}285$ ($\alpha = 3{\cdot}5$).

cells of closed streamlines with some downstream mass transfer occurring near the x axis. The amount of mass transfer diminishes as α increases from $3{\cdot}0$ to $3{\cdot}5$.

We now discuss the lee-wave pattern. The amplitude of the lee waves reaches a maximum for α between $2{\cdot}0$ and $2{\cdot}5$. The crest lines of the waves tilt to the left, becoming horizontal over the barrier and giving group velocities in the upstream direction, as expected. The flow on the 'upstream' face of the lee wave is in the upstream direction over the obstacle causing the flow there to appear as reversed jets. Further the flow appears to be blocked by the lee waves on the upstream side with negative velocities occurring far upstream of the wave for $\alpha = 2{\cdot}0$, $2{\cdot}1$ and $2{\cdot}5$. We may interpret the $\bar{\psi} = -2{\cdot}7$ streamline for $\alpha = 2{\cdot}1$ as a lee wave which has 'broken' in the upstream direction. The blocking caused by the lee waves tends to increase the upstream velocity at $z = 0$ for $\alpha = 1{\cdot}7$, $2{\cdot}0$ and $2{\cdot}1$ before the blocking column upstream of the obstacle is established.

A blocking column becomes established for some value of α between $2{\cdot}1$ and $2{\cdot}5$. No calculations were performed in this range as convergence quite unexpectedly and suddenly becomes extremely slow in this range. As α increases above $2{\cdot}5$ the blocking column upstream of the obstacle remains, with the lee waves diminishing in magnitude.

5. Discussion and comparison with earlier research

We may now consider the role of viscosity in determining our solution and then proceed to compare our solution with previous work.

We first note that use of a large but finite Reynolds number shows that the wavelike solution is a downstream solution, that $\alpha_3 = 0$ for $k < \alpha$ and that $\alpha_1 = 0$ for $k > \alpha$. Second, and most important, the inclusion of viscous forces requires that the no-slip condition be satisfied on the plate. This implies that ψ and

$\partial\psi/\partial x$ are continuous across $x = 0$ for all z and leads directly to the matching relations of (17) and (20), with no further assumptions required. This then eliminates the need for any *ad hoc* assumptions. The α_4 solution primarily acts to make the vertical velocity zero at the plate. Equation (17) implies that this boundary layer vanishes to lowest order for $z > 1$.

Third, a large but finite value for *Re* causes the perturbation to uniform flow to vanish far upstream which implies that reversed flows either form closed cells or S-shaped streamlines. Within closed cells the relation between ρ and ψ is indeterminate and the flow may be unstable. The flow along the upstream leg of the S-shaped streamlines is also unstable and local breakdown to turbulence may occur. Since the flow is statically stable outside these regions, the effects of breakdown may be localized.

Another question remains to be answered. After all this viscous analysis, have we not really just generated a solution to a linearized form of Long's equation? We can show that this is *not* the case as follows. Dropping the viscous term in (5) and integrating with respect to x, under the assumption that ψ vanishes far upstream, leads to the equation $\nabla^2\psi + \alpha^2\psi = 0$. Solutions to this equation satisfying $\psi = z$ on the plate can be obtained, see Miles (1968). Upon substituting our solution into this equation we can see that it *does not* satisfy the equation and hence our solution *is* a solution of the vorticity equation (5) but *not* of Long's equation. We return to this point later.

We now show that the solution obtained by Trustrum (1964, 1971) for the density and pressure field is drastically altered if a small viscosity is retained in the transient analysis. We shall merely sketch the procedure for an unbounded flow, since the analysis is quite similar to that of Trustrum. An initially uniform, linearly stratified flow is disturbed at $t = 0$ by a disturbance on the plane $x = 0$. We first replace (1 *e*) by

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x} = \frac{\partial\psi}{\partial x}, \tag{28}$$

where the density perturbation has been non-dimensionalized with respect to $\rho_0\beta b$ and time with respect to b/U. The appropriate transient terms are also added to (1 *a*, *b*). We then take the Laplace transform of the resulting transient equations and, following Trustrum, seek solutions of the form

$$\bar{\rho}(x,z,s) = \frac{R(k,s)}{s}\sin kz\, e^{\alpha(k,s)x}, \tag{29a}$$

$$\bar{\psi}(x,z,s) = \frac{\psi(k,s)}{s}\sin kz\, e^{\alpha(k,s)x}, \tag{29b}$$

where s is the Laplace transform variable ($\mathrm{Re}\, s > 0$). Equation (28) leads to

$$R(k,s) = \frac{\alpha}{s+\alpha}\psi(k,s). \tag{30}$$

If $\alpha(k, 0)$ does not vanish, then, in the steady state, (30) implies that

$$\rho(x,z) = \psi(x,z).$$

The transient analogue of ($8c$) leads to an algebraic equation for $\alpha(k,s)$. With $Re^{-1} = 0$ in that equation, Trustrum finds two solutions for α which go to zero with s. First, for small s, $\alpha = \alpha_- = -F_i ks/(1+F_i k)$. For this downstream solution $\rho_-(k,z) = -F_i k\psi_-(k,0)\sin kz$. This is the additional downstream solution, mentioned earlier, obtained by Trustrum and is associated with the coefficients B_n in the latter paper. For $Re^{-1} \neq 0$ and $s < k^4/Re$ the transient equation shows that

$$\alpha_- \cong -s + sF_i^{-2}(sRe/k^4).$$

This implies that for small s

$$\rho_-(k,s) = -\frac{k^4F_i^2}{Re}\frac{\psi_-(k,s)}{s}.$$

For the density perturbation to remain finite in the steady state $\psi_-(k,0) = 0$. This additional downstream solution corresponds to an arbitrary $\rho_-(z)$ and $p_-(z)$ with no associated velocity field in the steady state. Thus B_n should be set equal to zero in equation (2.2) of Trustrum (1971). With $Re^{-1} = 0$, the second solution for α which goes to zero with s is, for small s, $\alpha_+ = F_i ks/(1-F_i k)$. The corresponding density field has $\rho_+(k,z) = F_i k\psi_+(k,0)\sin kz$. This is an upstream solution for $kF_i < 1$, and a downstream solution for $kF_i > 1$. We can identify this root with our α_3 for $kF_i < 1$ and with α_1 for $kF_i > 1$ if $Re^{-1} = 0$. However, if $Re^{-1} \neq 0$, we recall that α_3 ($kF_i < 1$) and α_1 ($kF_i > 1$) are of $O(Re^{-1})$. If $Re^{-1} \neq 0$, the transient equation for α_+ shows that for small s

$$\alpha_+ = \frac{F_i ks}{1-F_i k} + \operatorname{sgn}(1-F_i k)\frac{g(k)}{Re},$$

where $g(k) > 0$. This solution, therefore, *does not vanish with* s which implies that for this solution $\rho_+(k,z) = \psi_+(k,0)\sin kz$ ($\rho_+(z) = \psi_+(z)$). Thus, the relation between the density (or pressure) field and the stream function is drastically altered if $Re^{-1} \neq 0$. In equations (2.3) and (2.4) of Trustrum (1971) we should replace C_n with $(k/n)C_n$. Since the matching conditions were partially formulated in terms of the density and pressure fields, the numerical results of that paper are questionable. We have now shown that for all but the B_n solution the density perturbation equals the stream function perturbation. Requiring that the total stream function and density be continuous above the plate implies that $\rho_-(z) = 0$ for $z > 1$. Thus the additional B_n solution is apparently only necessary if some condition on the density field is specified on the plate which is inconsistent with $\rho = \psi$. Thus, case B considered by Trustrum, with $C_n \to (k/n)C_n$, is analogous to the problem considered here, and the matching conditions can also be shown to be analogous.

The second study we shall consider is that of Miles (1968, 1971). As a part of the earlier paper, Miles considers stratified flow over a vertical barrier in a half-space. Using Long's equation with the displacement function as the dependent variable, he first transforms from rectangular to elliptic co-ordinates and then obtains the solution in terms of a series of products of Mathieu functions. He then computes the drag coefficient of the barrier, and notes that reversed flow occurs in the lee-wave field for $\alpha \geqslant 1{\cdot}73$, which possibly limits the applicability of Long's equation. He does not compute streamline patterns for various α.

Our calculations show some slight reversed flow in the lee-wave field. In his latter paper, Miles notes that reversed flow at the upstream stagnation point $(0^-, 0)$ occurs for $\alpha \geqslant 2{\cdot}05$, while we concluded that this occurs for $\alpha \geqslant 2{\cdot}10$. We note that our calculations show upstream reversed flow 'induced' by the lee-wave field, near $z = 2$, for $\alpha \geqslant 2{\cdot}0$. Since Miles has calculated no streamline pattern we cannot make any further comparisons. As we have shown, our solution is not a solution of Long's equation but may be regarded as an alternative to Miles' solutions. The reason for this is as follows. Miles solution to Long's equation follows directly from the integration of the vorticity equation along streamlines with the function of integration evaluated far upstream under the *assumption* of undisturbed flow in that region. The inviscid limit of our solution reflects, although in linearized form, the integration of the vorticity equation along streamlines with the function of integration evaluated in the plane of the plate and determined by boundary and matching conditions applied there. This procedure is more reasonable on physical grounds since the presence of the plate does determine the nature of the disturbance. Further, there is no theoretical reason for the functions of integration to coincide. Our results, however, do show that for $\alpha \leqslant 1{\cdot}7$ there is relatively little upstream disturbance, and so there should be reasonable agreement between Miles's results and those obtained here.

6. Conclusions

We have obtained the predictions of the Oseen model for a range of internal Froude numbers and have noted that, at best, they might yield a qualitative picture of the actual flow field.

We note that an analogy exists, under certain conditions, between two-dimensional stratified flows and the horizontal flow of a homogeneous fluid over a flat bottom in the 'beta plane', see Martin (1966). The analogy with the solutions obtained here is as follows. An eastwards homogeneous current of speed U flows over a flat bottom towards a narrow trench or ridge of width $2b$ running in the north–south direction. The rapid change of depth would tend to cause the current to flow around rather than across the trench. The trench then is the analogue of the plate with $\alpha^2 \equiv b^2 U^{-1} df/dy$. Our solutions might then apply to an oceanic problem.

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REFERENCES

Bretherton, F. P. 1967 The time-dependent motion due to a cylinder in an unbounded rotating or stratified fluid. *J. Fluid Mech.* **28**, 545.

Janowitz, G. S. 1968 On wakes in stratified fluids. *J. Fluid Mech.* **33**, 417.

Janowitz, G. S. 1971 The slow transverse motion of a flat plate through a non-diffusive stratified fluid. *J. Fluid Mech.* **47**, 171.

Long, R. R. 1953 Some aspects of the flow of stratified fluids. I. A theoretical investigation. *Tellus*, **5**, 42.

MARTIN, S. 1966 The slow motion of a finite flat plate through a viscous stratified fluid. *Dept. of Mech., The Johns Hopkins University Tech. Rep.* no. 21 (*ONR Ser.*) Nonr-4010(01).

MILES, J. W. 1968 Lee waves in a stratified flow. Part 1. Thin barrier. *J. Fluid Mech.* **32**, 549.

MILES, J. W. 1971 Upstream boundary-layer separation in stratified flow. *J. Fluid Mech.* **48**, 791.

TRUSTRUM, K. 1964 Rotating and stratified fluid flow. *J. Fluid Mech.* **19**, 415.

TRUSTRUM, K. 1971 An Oseen model of the two-dimensional flow of a stratified fluid over an obstacle. *J. Fluid Mech.* **50**, 177.